PQ. Recall the following fact that you proved on the first homework. If $\sum_{n=0}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Which of the following statements is equivalent to this fact?

(a) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{i=0}^{\infty} a_n$ converges

(b) If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{i=0}^{\infty} a_n$ diverges.

(c) If $\sum_{i=0}^{\infty} a_n$ diverges, then $\lim_{n\to\infty} a_n \neq 0$.

Thm: (The Divergence Test) If the sequence of terms in a series do not converge to 0, then the series diverges.

Q. If you are asked to determine whether a series converges or diverges, when should you use this test?

First! This is an easy thing to check

Ex 1 Does the series $\sum_{n=1}^{\infty} \cos(1/n)$ converge or diverge?

$$\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos\left(\lim_{n\to\infty} \frac{1}{n}\right) = \cos(0) = 1$$

$\longrightarrow \sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$ diverges

Ex 2 Does $\sum_{n=1}^{\infty} \frac{1}{n}$ converge or diverge?

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

$\longrightarrow$ no conclusion

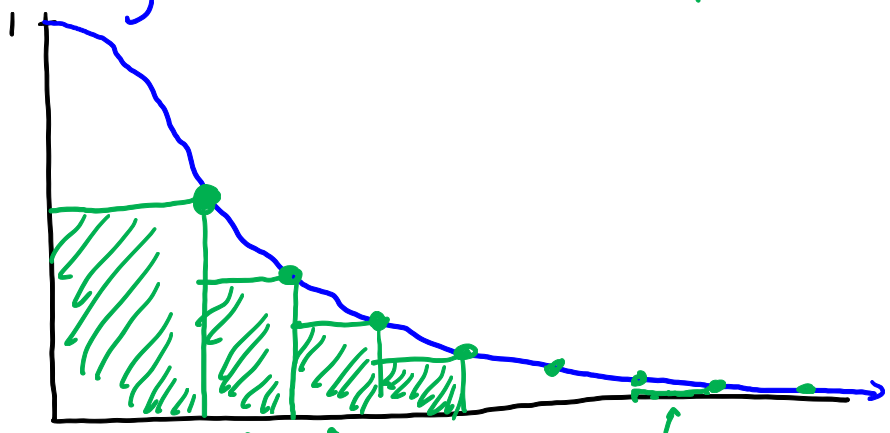
Note (IMPORTANT) If $\lim_{n\to\infty} a_n = 0$, the divergence test tells you nothing.

Q. How can we relate series to things that we already understand? Like the series $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$?

$y = \frac{1}{1+x^2}$ $\quad a_n = \frac{1}{1+n^2}$

$n=1$

$= \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \cdots$



Area = b · h

$= 1 \cdot \frac{1}{1+1^2}$ $\quad 1 \cdot \frac{1}{1+2^2}$ $\quad 1 \cdot \frac{1}{1+3^2}$ $\quad 1 \cdot \frac{1}{1+n^2}$

$\frac{1}{2}$ $\quad \frac{1}{5}$ $\quad \frac{1}{10}$ $\quad a_n$

Total green area $= \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \cdots + \frac{1}{1+n^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{1+n^2}$

$\int_0^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \arctan(x)\Big|_0^t$

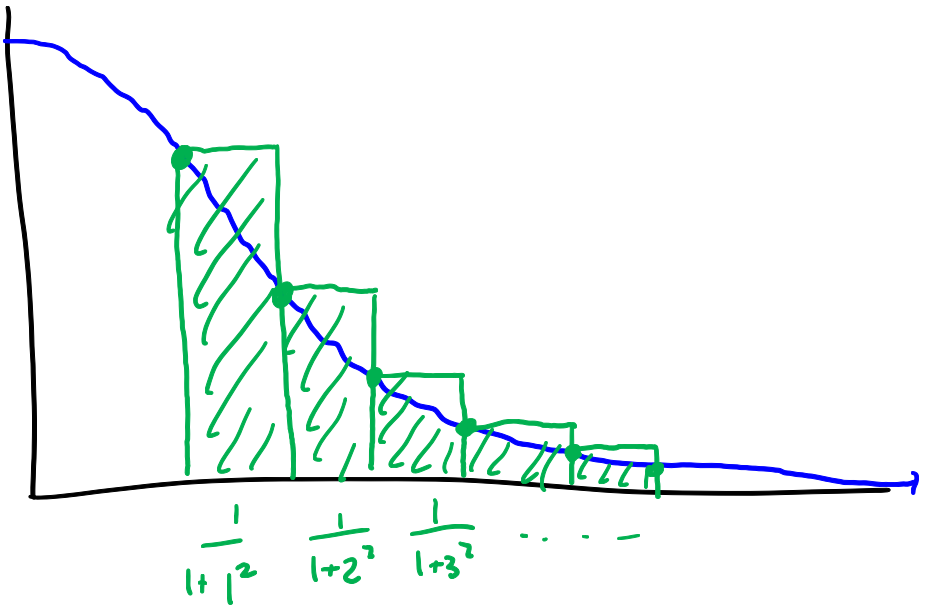
$= \lim_{t\to\infty} \arctan(t) - \arctan(0) = \frac{\pi}{2}$

Conclusion: $\sum_{n=1}^{\infty} \frac{1}{1+n^2} \le \int_0^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$

Note: $S_k = \frac{1}{1+1^2} + \cdots + \frac{1}{1+k^2} \le \frac{1}{1+1^2} + \cdots + \frac{1}{1+k^2} + \frac{1}{1+(k+1)^2} = S_{k+1}$

so $\{S_k\}$ is monotone and bounded. Hence $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ converges

Q. What is $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$?



$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \text{Total green area} \geq \int_1^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to\infty} \arctan(t) - \arctan(1)$$

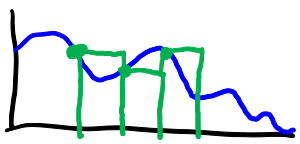
$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

So $$\frac{\pi}{4} = \int_1^{\infty} \frac{1}{1+x^2}\,dx \leq \sum_{n=1}^{\infty} \frac{1}{1+n^2} \leq \int_0^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$$

Note: We used the fact that

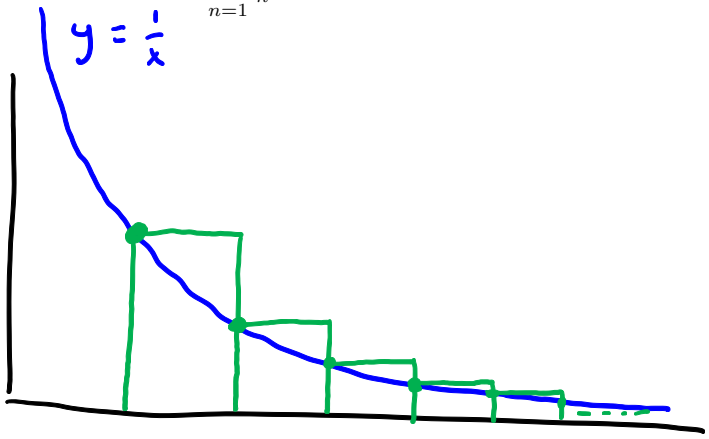
(1) $\frac{1}{1+x^2} \geq 0$ (in MCT)

(2) $\frac{1}{1+x^2}$ decreasing (with the inequalities)



← will $\int$ be over- or underestimate?

Ex 3 Decide whether $\sum_{n=1}^{\infty} \frac{1}{n}$ converges or diverges.



$\sum_{n=1}^{\infty} \frac{1}{n}$ = Total green area $\geq \int_1^{\infty} \frac{1}{x}\,dx$

$= \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$

$= \lim_{t\to\infty} \ln|x| \Big|_1^t$

$= \lim_{t\to\infty} \ln|t| - \ln|1|$

$= \infty$

$\longrightarrow \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

Thm: (The Integral Test) Suppose $\sum_{n=1}^{\infty} a_n$ is a series with positive terms. Suppose there is a function f and a positive integer n_0 such that the following conditions are met:

(a) f is continuous for all $x \geqslant n_0$

(b) f is decreasing for all $x \geqslant n_0$

(c) $f(n) = a_n$ for all integers $n \geqslant n_0$ (ie. f interpolates $\{a_n\}$ for all $n \geqslant N$)

Then

- If $\int_N^{\infty} f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\int_N^{\infty} f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Ex: for $a_n = \frac{1}{1+n^2}$, $f(x) = \frac{1}{1+x^2}$

None: for $a_n = (-1)^n$, $f(x) = (-1)^x$ doesn't make sense

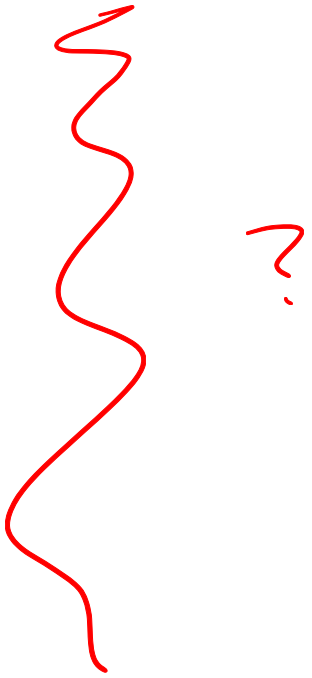
Q. What is great about this theorem?

- Gives a test for convergence and divergence simultaneously

Q. What is not great about this theorem?

- Integrals are hard!
- $f(x)$ doesn't always exist
- Some series have negative terms or don't decrease

Def: If p is a real number, we call the series $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$ a p-series.



Q. For which values of p does the p-series converge?

$p < 0$ (think: $p = -1$)

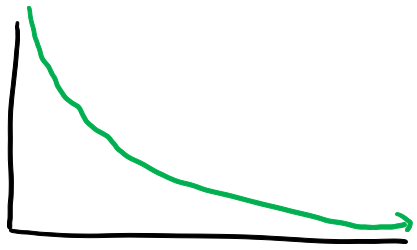
$\sum_{n=1}^{\infty} \frac{1}{n^{-1}} = \sum_{n=1}^{\infty} n$ diverges (since terms $\to \infty$)

$p = 0$

$\sum_{n=1}^{\infty} \frac{1}{n^0} = \sum_{n=1}^{\infty} 1$ diverges (since summands $\to 1$)

$p > 0, p \neq 1$

$\sum_{n=1}^{\infty} \frac{1}{n^p} \rightsquigarrow f(x) = \frac{1}{x^p}$



cts. ✓

dec. ✓

$f(n) = \frac{1}{n^p}$ ✓

$f(x) \geq 0$ ✓

Apply Integral Test:

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ Converges if and only if $\int_1^{\infty} \frac{1}{x^p}\,dx$ Converges

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t x^{-p}\,dx = \lim_{t\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^t = \lim_{t\to\infty} \frac{t^{1-p}}{1-p} - \frac{1}{1-p}$$

$$= \begin{cases} \infty & 0 < p < 1 \\ \frac{1}{p-1} & 1 < p \end{cases}$$

Conclude: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\begin{cases} \text{conv.} & p > 1 \\ \text{div.} & 0 < p < 1 \end{cases}$

$p = 1$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

Conclusion: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\begin{cases} \text{diverges} & p \leq 1 \\ \text{converges} & p > 1 \end{cases}$

Ex 4 Decide whether the series $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots$ converges or not.

(a) Converges
(b) Diverges
(c) Neither

Q. We know that if $p > 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^p} = L$ for *some* L. Can we find the value of L?

$S_k = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{k^p} = ?$

$L = \lim_{k\to\infty} S_k = ?$

Fact: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Thm: (Integral Test Remainder Estimate) Suppose $\sum_{n=1}^{\infty} a_n$ is a series with positive terms. Suppose there is a function f and a positive integer ~~n_0~~ k such that the following conditions are met:

(a) f is continuous for $x \geqslant$ ~~n_0~~
(b) f is decreasing for $x \geqslant$ ~~n_0~~
(c) $f(n) = a_n$ for all integers $n \geqslant$ ~~n_0~~.

Then if $\sum_{n=1}^{\infty} a_n$ converges, we get a remainder estimate:

(do this 2nd) $\int_{k+1}^{\infty} f(x)\,dx \leq R_k \leq \int_{k}^{\infty} f(x)\,dx$

(do this first)

As usual, $S_k = a_1 + \cdots + a_k$

If $\sum_{n=1}^{\infty} a_n = L$, then $S_k \approx L$

$\rightarrow \quad L - S_k \approx 0$

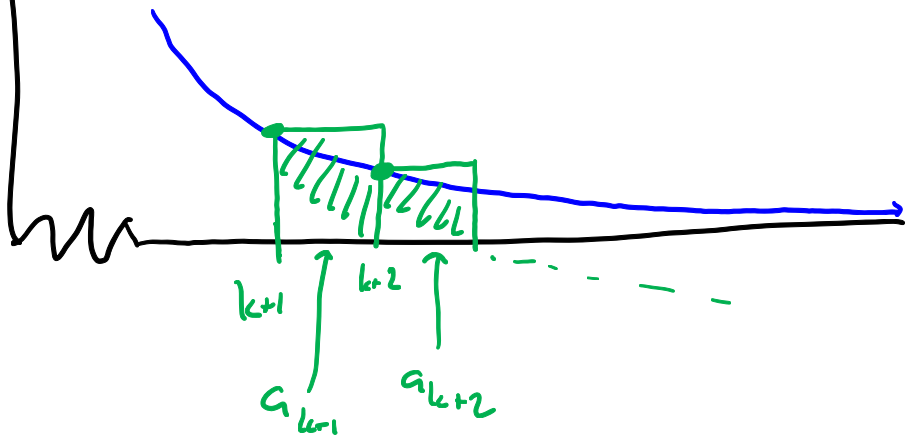
$\rightarrow \quad \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{k} a_n \approx 0$

$\rightarrow \quad (a_1 + a_2 + a_3 + \cdots) - (a_1 + a_2 + \cdots + a_k) \approx 0$

$\rightarrow \quad a_{k+1} + a_{k+2} + \cdots \approx 0$

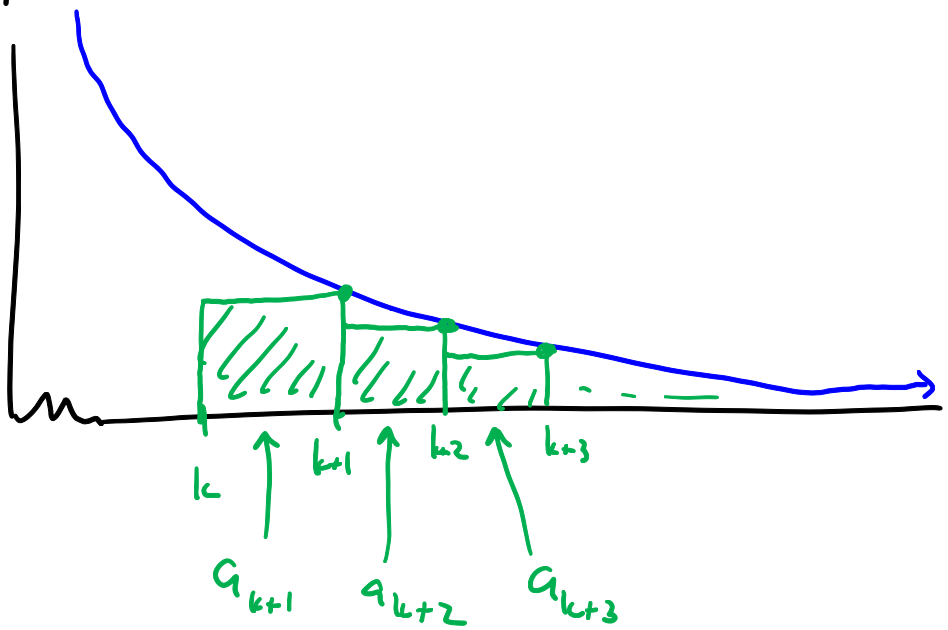
$R_k = \sum_{n=k+1}^{\infty} a_n$ ← remainder OR error

Lower bound



$$\int_{k+1}^{\infty} f(x)\,dx \leq R_k = a_{k+1} + a_{k+2} + \cdots$$

Upper bound



$$R_k = \sum_{n=k+1}^{\infty} a_{k+1} \leq \int_k^{\infty} f(x)\,dx$$

Ex 5 Consider $\sum_{n=0}^{\infty} \frac{1}{1+n^2}$.

(a) Find explicit upper and lower bounds for $R_N = \sum_{n=N+1}^{\infty} \frac{1}{1+n^2}$ in terms of N.

$$\int_{k+1}^{\infty} \frac{1}{1+x^2}\,dx \leq R_k \leq \int_{k}^{\infty} \frac{1}{1+x^2}\,dx$$

$$\frac{\pi}{2} - \arctan(k+1) \leq R_k \leq \frac{\pi}{2} - \arctan(k)$$

(b) Use the partial sum $s_{1000} = \sum_{n=1}^{1000} \frac{1}{1+n^2}$ to estimate the value of the "full" series $L = \sum_{n=1}^{\infty} \frac{1}{1+n^2}$, and give a bound for the error.

Computer: $S_{1000} \approx 1.076$

Error: $R_{1000} \leq \frac{\pi}{2} - \arctan(1000) \approx 0.001$

$\rightarrow 1.075 \leq \sum_{n=1}^{\infty} \frac{1}{1+n^2} \leq 1.077$

(c) Repeat (b) with s_{100000}.

Computer: $S_{100000} \approx 1.0766040$

$R_{100000} \leq \frac{\pi}{2} - \arctan(100000) \approx 0.00001$

$\rightarrow 1.07659 \leq \sum_{n=1}^{\infty} \frac{1}{1+n^2} \leq 1.0768$

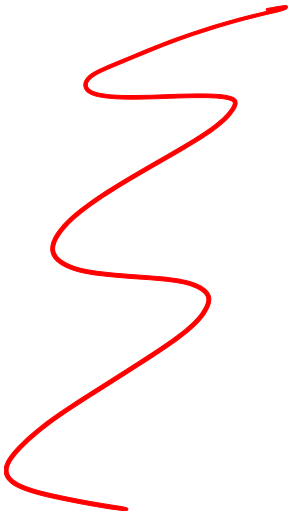
Exact: $\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{1 + e^{2\pi}(\pi - 1) + \pi}{2(e^{2\pi} - 1)}$

Ex 6 Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^5}$.

(a) Why does the series converge?

p-series with $p=5>1$

(b) Why is it hard to find the exact value of $L = \sum_{n=1}^{\infty} \frac{1}{n^5}$?



$S_k = 1 + \frac{1}{2^5} + \frac{1}{3^5} + \dots + \frac{1}{k^5}$

doesn't simplify nicely

(c) Estimate L with error $< 10^{-5}$.

Integral test applies (we already checked this), so remainder estimate applies

Want: $R_k < 10^{-5}$

Since $R_k \le \int_k^{\infty} \frac{1}{x^5}\,dx$, if $\int_k^{\infty} \frac{1}{x^5}\,dx < 10^{-5}$, R_k also $< 10^{-5}$

Goal: $\int_k^{\infty} x^{-5}\,dx < 10^{-5}$

$$\int_k^\infty x^{-5}\,dx = \lim_{t\to\infty} \int_k^t x^{-5}\,dx = \lim_{t\to\infty} \frac{x^{-4}}{-4}\Big|_k^t$$

$$= \lim_{t\to\infty} \frac{1}{-4t^4} - \frac{1}{-4k^4} = \frac{1}{4k^4}$$

Goal: $\frac{1}{4k^4} < 10^{-5} \longleftrightarrow \frac{10^5}{4} < k^4$

$$\longleftrightarrow \left(\frac{10^5}{4}\right)^{1/4} < k$$

$$\sqrt[4]{}$$

$$12.57$$

$$\longrightarrow k = 13$$

Get: $R_{13} < 10^{-5}$

Conclude: $L \approx S_{13} = \sum_{n=1}^{13} \frac{1}{n^5} \approx 1.03692$

with error < 0.00001

great least as many digits as error term has